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Bethe ansatz for spin-1 chain with long-range interaction

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Abstract. Fusion procedure for the R-matrices with disorder parameter μ_i and the power-series expansion technique of de Vega have been used to construct a spin-1 chain with long-range interaction. The energy eigenvalue and the equation for the momenta for the state containing m-excitations are determined with the help of the algebraic Bethe ansatz.

1. Introduction

Studies of spin chain greater than 1/2 became a reality with the advent of the fusion technique for R-matrices [1]. Though Zamoldochikov and Fateev [2] made the initial attempt and constructed the R-matrix for the spin-1 system by solving the Yang–Baxter equation, the complete solution could not be obtained until the method of fusion was adopted. The explicit construction of the Bethe ansatz was obtained by using the commutation rules dictated by the 9×9 R-matrix and by using the elegant formalism of the fusion procedure [3].

In this paper we show how the fusion procedure can be combined with the methodology suggested by de Vega [4] to construct a spin-1 chain with long-range interaction (that is not restricted to the nearest neighbour). Due to the methodology of the fusion technique we have been able to set up the Algebraic Bethe ansatz for the mth excitation state. The energy eigenvalue of such a state and the equation determining the m-eigenmomenta are explicitly deduced.

Our paper is organised as follows. In section 2, for completeness, we have re-derived the Bethe ansatz results for the usual xxz case which we will use in the subsequent sections. In section 3 we construct the first and second fused R-matrices. In section 4 we show how the long-range Hamiltonian involving spin-1 operators can be constructed. Finally in sections 5 and 6 we deduce the corresponding eigenvalues and the Bethe ansatz equations.

2. Formulation

To begin let us consider a Heisenberg spin chain with nearest neighbour interaction (spin of each atom at the lattice site is equal to 1/2) governed by the quantum R-matrix [5]

$$R(\theta) = \begin{bmatrix} \sin(\theta+\eta) & 0 & 0 & 0 \\ 0 & \sinh\theta & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh\theta & 0 \\ 0 & 0 & 0 & \sinh(\theta+\eta) \end{bmatrix}. \tag{1}$$

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The basic observation of [4] is that, even if one introduces inhomogeneities $\mu_1, \mu_2, \ldots, \mu_n$ at the lattice sites $i = 1, 2, \ldots, N$, the model remains integrable and the systems still possess an infinite number of integrals of motion in involution.

In general, a monodromy operator $T(\theta, \{\mu\})$ is written as

$$T^\sigma(\theta, \{\mu\}) = \prod_{i=1}^{N} R^\sigma_{0i}(\theta + \mu_i)$$

where R_{0i} is defined over $V_0 \otimes V_i$, V_0 being the quantum space, and σ is used to denote the purely spin-1/2 character of the R matrix. Also

$$T^\sigma(\theta, \{\mu\}) = \begin{pmatrix} A^\sigma(\theta, \{\mu\}) & B^\sigma(\theta, \{\mu\}) \\ C^\sigma(\theta, \{\mu\}) & D^\sigma(\theta, \{\mu\}) \end{pmatrix}.$$

The corresponding transfer matrix $t^\sigma(\theta, \mu)$ is

$$t^\sigma(\theta, \{\mu\}) = \mathrm{Tr}_0\, T^\sigma(\theta, \{\mu\}).$$

The matrix element of $T(\theta, \{\mu\})$ will be denoted as $T_{\alpha\beta}(\theta, \{\mu\})$. The above assertion implies that

$$[\mathrm{Tr}_0\, T(\theta, \{\mu\}), \mathrm{Tr}_0\, T(\theta, \{\mu\})] = 0 \tag{3}$$

where $T(\theta, \mu)$ denotes the monodromy operator for the inhomogeneous model and Tr_0 denotes that the operation of taking the trace is to be performed over the quantum space.

The monodromy matrix for the inhomogeneous model reads in terms of site operators [4] as

$$T_{ab}(\theta, \{\mu\}) = \sum_{a_1, a_2, \ldots, a_{N-1}=1}^{2} t^{(1)}_{aa_1}(\theta + \mu_1) t^{(2)}_{a_1 a_2}(\theta + \mu_2) \ldots t^{(N)}(\theta + \mu_N) a_{N-1} b. \tag{4}$$

The operators $t^{(k)}(\theta + \mu_k) a_{k-1} a_k$ act on the 2D vertical space $V^{(k)}$. We have

$$\begin{aligned} t(\theta)_{11} &= \begin{pmatrix} \sinh(\theta + \eta) & 0 \\ 0 & \sinh\theta \end{pmatrix} \qquad t(\theta)_{12} = \sigma_- \sinh\eta \\ t(\theta)_{22} &= \begin{pmatrix} \sinh\theta & 0 \\ 0 & \sinh(\theta + \eta) \end{pmatrix} \qquad t(\theta)_{21} = \sigma_+ \sinh\eta \end{aligned} \tag{5}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Such a monodromy matrix also satisfies

$$R^\sigma(\theta - \theta') T^\sigma(\theta, \{\mu\}) T^\sigma(\theta', \{\mu\}) = T^\sigma(\theta', \{\mu\}) T^\sigma(\theta, \{\mu\}) R^\sigma(\theta - \theta') \tag{6}$$

with R given by (1).

From the Yang–Baxter equation (6) we obtain

$$A^{\sigma}(\theta)B^{\sigma}(\theta_1) = \frac{\sinh(\theta_1 - \theta + \eta)}{\sinh(\theta_1 - \theta)}B^{\sigma}(\theta_1)A^{\sigma}(\theta) - \frac{\sinh\eta}{\sinh(\theta_1 - \theta)}B^{\sigma}(\theta)A^{\sigma}(\theta_1) \tag{7}$$

$$D^{\sigma}(\theta)B^{\sigma}(\theta_1) = \frac{\sinh(\theta - \theta_1 + \eta)}{\sinh(\theta - \theta_1)}B^{\sigma}(\theta_1)D^{\sigma}(\theta) - \frac{\sinh\eta}{\sinh(\theta - \theta_1)}B^{\sigma}(\theta)D^{\sigma}(\theta_1) \tag{8}$$

$$C^{\sigma}(\theta)B^{\sigma}(\theta_1) = B^{\sigma}(\theta_1)C^{\sigma}(\theta) + \frac{\sinh\eta}{\sinh(\theta_1 - \theta)}[D^{\sigma}(\theta_1)A^{\sigma}(\theta) - D^{\sigma}(\theta)A^{\sigma}(\theta_1)]. \tag{9}$$

We shall treat B^{σ} as the creation operator and C^{σ} as the destruction operator.

Let us start with a single particle state $|\Omega_1\rangle = B(\mathrm{v}_1)|0\rangle_{\sigma}$. It may be noted that the mth particle state is given as

$$|\Omega_m\rangle = B^{\sigma}(\mathrm{v}_1)B^{\sigma}(\mathrm{v}_2)\dots B^{\sigma}(\mathrm{v}_m)|0\rangle_{\sigma} \tag{10}$$

and that $C^{\sigma}(\theta)$ is the destruction operator, $C^{\sigma}(\theta)|0\rangle_{\sigma} = 0$. Now consider

$$[A^{\sigma}(\theta,\{\mu\}) + D^{\sigma}(\theta,\{\mu\})]|\Omega_m\rangle = \alpha(u)\prod_{i=1}^{m}\frac{\sinh(v_i - \theta + \eta)}{\sinh(v_i - \theta)} + \delta(u)\prod_{i=1}^{m}\frac{\sinh(\theta - v_i + \eta)}{\sinh(\theta - v_i)} \tag{11}$$

where

$$\begin{aligned} A^{\sigma}(\theta,\{\mu\})|\Omega_m\rangle &= \alpha(u)|\Omega_m\rangle \\ D^{\sigma}(\theta,\{\mu\})|\Omega_m\rangle &= \delta(u)|\Omega_m\rangle. \end{aligned} \tag{12}$$

$\alpha(u)$, $\delta(u)$ are given by

$$\begin{aligned} \alpha(\theta,\{\mu\}) &= \prod_{k=1}^{m}\sinh(\theta + \mu_k + \eta) \\ \delta(\theta,\{\mu\}) &= \prod_{k=1}^{m}\sinh(\theta + \mu_k). \end{aligned} \tag{13}$$

In deriving these expressions we have utilised expressions (4) and (5) for T_{ab} and the fact that

$$|0\rangle_{\sigma} = \prod_{i=1}^{N}\otimes\begin{pmatrix}1\\0\end{pmatrix}_i. \tag{14}$$

It is not difficult to deduce that the eigenvalue of such an excited state is given as

$$E^{\sigma}(\theta) = \prod_{i=1}^{m}\frac{\sinh(v_i - \theta + \eta)}{\sinh(v_i - \theta)}\alpha(u) + \prod_{i=1}^{m}\frac{\sinh(\theta - v_i + \eta)}{\sinh(\theta - v_i)}\delta(u). \tag{15}$$

The eigenmomenta v_i are given by the equations obtained by equating the residues at the poles of $E^{\sigma}(\theta)$ to zero. In this way one obtains

$$\prod_{k=1}^{m}\frac{\sinh(v_j + \mu_k + \eta)}{\sinh(v_j + \mu_k)} = \prod_{\substack{i=1\\ i\neq j}}^{m}\frac{\sinh(v_i - v_j - \eta)}{\sinh(v_i - v_j + \eta)}. \tag{16}$$

Here we have actually re-derived the basics of a spin-1/2 chain in a slightly different way; we will be referring to these repeatedly in what follows.

3. The fusion process

We now fuse two spin-1/2 quantum R^{σ} matrices to construct an intermediate R^{σ_s} matrix:

$$R^{\sigma_s}(\theta, \{\mu\}) = P_{12}^{+} R^{\sigma}(\theta - \eta/2) R^{\sigma}(\theta + \eta/2) P_{12}^{+} \tag{17}$$

where the suffix s denotes that half of the spin states are now actually spin-1 due to the fusion procedure. The corresponding monodromy matrix is now

$$T^{\sigma_s}(\theta, \{\mu\}) = P_{12}^{+} T_1^{\sigma}(\theta - \eta/2, \{\mu\}) T_2^{\sigma}(\theta + \eta/2, \{\mu\}) P_{12}^{+} \tag{18}$$

where P_{12}^{+} is the projection operator.

The transfer matrix for the mixed σ_s situation is given by

$$\begin{aligned} t^{\sigma_s}(\theta, \{\mu\}) &= \mathrm{Tr}_{12}[P_{12}^{+} T_1^{\sigma}(\theta - \eta/2, \{\mu\}) T_2^{\sigma}(\sigma + \eta/2, \{\mu\}) P_{12}^{+}] \\ &= t^{\sigma}(\theta + \eta/2, \{\mu\}) t^{\sigma}(\theta + \eta/2, \{\mu\}) \\ &\quad - \mathrm{Tr}_{12}[P_{12}^{-} T_1^{\sigma}(\theta + \eta/2, \{\mu\}) T_2^{\sigma}(\theta + \eta/2, \{\mu\})]. \end{aligned} \tag{19}$$

The last term is just the quantum determinant

$$\Delta(\theta) = \mathrm{Tr}\left[\frac{1-P}{2} T_1^{\sigma}(\theta - \eta/2, \{\mu\}) T_2^{\sigma}(\theta + \eta/2, \{\mu\})\right] \tag{20}$$

where P is the permutation operator.

Explicitly, we have

$$\Delta(\theta) = A^{\sigma}(\theta + \eta/2, \{\mu\}) D^{\sigma}(\theta - \eta/2, \{\mu\}) - B^{\sigma}(\theta + \eta/2, \{\mu\}) C^{\sigma}(\theta - \eta/2, \{\mu\}). \tag{21}$$

Eigenvalues and Eigenvectors of the mixed transfer matrix t^{σ_s} can be obtained from the crucial observation that t^{σ_s} and t^{σ} commute:

$$[t^{\sigma_s}(\theta, \{\mu\}), t^{\sigma}(\theta, \{\mu\})] = 0 \tag{22}$$

so that they do possess common eigenvectors. So using $|\Omega_m\rangle$, found in the previous section, we can at once obtain the eigenvalue of t^{σ_s} as

$$\begin{aligned} E_m^{\sigma_s}(\theta) &= E_m^{\sigma}(\theta - \eta/2) E_m^{\sigma}(\theta + \eta/2) - d(\theta) \\ d(\theta) &= \prod_{k=1}^{m} \frac{\sinh(v_k - \theta + \eta/2)}{\sinh(v_k - \theta - \eta/2)} \prod_{l=1}^{m} \frac{\sinh(\theta - v_l + \eta/2)}{\sinh(\theta - v_l - \eta/2)} \alpha(\theta + \eta/2) \delta(\theta - \eta/2) \\ &= \alpha(\theta + \eta/2) \delta(\theta - \eta/2). \end{aligned} \tag{23}$$

We now perform the second fusion to obtain the full spin-1 chain. We denote the corresponding monodromy matrix as T^{s}:

$$T^{s}(\theta, \{\mu\}) = P_{12}^{+} T_1^{\sigma_s}(\theta - \eta/2, \{\mu\}) T_2^{\sigma_s}(\theta + \eta/2, \{\mu\}) P_{12}^{+}. \tag{24}$$

Taking the trace of both sides we obtain

$$\begin{aligned} t^{s}(\theta, \{\mu\}) &= \mathrm{Tr}[P_{12}^{+} T_1^{\sigma_s}(\theta - \eta/2, \{\mu\}) T_2^{\sigma_s}(\theta + \eta/2, \{\mu\}) P_{12}^{+}] \\ &= t^{\sigma_s}(\theta - \eta/2, \{\mu\}) t^{\sigma_s}(\theta - \eta/2, \{\mu\}) - \Delta(\theta). \end{aligned} \tag{25}$$

where $\Delta(\theta)$ is again a quantum determinant given as

$$\begin{aligned}\Delta(\theta) &= \text{q-det}\, T_1^{\sigma_s}(\theta - \eta/2)\, \text{q-det}\, T_2^{\sigma_s}(\theta + \eta/2) \\ &= \Delta(\theta - \eta/2)\Delta(\theta + \eta/2).\end{aligned} \tag{26}$$

From equation (19) we now obtain

$$\begin{aligned}t^s(\theta, \{\mu\}) &= [t^\sigma(\theta-\eta, \{\mu\})t^\sigma(\theta, \{\mu\}) - \Delta(\theta-\eta/2)][t^\sigma(\theta, \{\mu\})t^\sigma(\theta+\eta, \{\mu\}) - \Delta(\theta+\eta/2)] \\ &\quad - \Delta(\theta - \eta/2)\Delta(\theta + \eta/2) \\ &= t^\sigma(\theta - \eta, \{\mu\})t^\sigma(\theta, \{\mu\})t^\sigma(\theta, \{\mu\})t^\sigma(\theta + \eta, \{\mu\}) \\ &\quad - t^\sigma(\theta-\eta, \{\mu\})t^\sigma(\theta, \{\mu\})\Delta(\theta+\eta/2) - \Delta(\theta-\eta/2)t^\sigma(\theta, \{\mu\})t^\sigma(\theta+\eta, \{\mu\}).\end{aligned} \tag{27}$$

Again, with the help of the Yang–Baxter equation we can prove that

$$[t^s(\theta, \{\mu\}), t^{\sigma_s}(\theta, \{\mu\})] = 0 \tag{28}$$

so that they also have common eigenvectors. This was the main component in the analysis of [3]. So, if we operate with t^s, as given in equation (27), on $|\Omega_m\rangle$ we obtain

$$\begin{aligned}E_m^s(\theta) &= E_m^\sigma(\theta - \eta)E_m^\sigma(\theta)E_m^\sigma(\theta)E_m^\sigma(\theta + \eta) - E_m^\sigma(\theta - \eta)E_m^\sigma(\theta)d(\theta + \eta/2) \\ &\quad - d(\theta - \eta/2)E_m^\sigma(\theta)E_m^\sigma(\theta + \eta).\end{aligned} \tag{29}$$

We have now deduced the general form of the eigenvalues of the mth excited state for a spin-1 system by the technique of double fusion, without any reference to the specific form of the Hamiltonian.

In the following section we will first deduce the long-range interaction Hamiltonian and then extract the corresponding eigenvalue from the general expression (29).

4. Long-range Hamiltonian

We start with the one-time fused form of the R matrix $R^{\sigma_s}(\omega)$:

$$R^{\sigma_s}(\omega) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & d & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 \\ 0 & d & 0 & c & 0 & 0 \\ 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

with

$$a(\omega) = \sinh(\omega + \tfrac{3}{2}\eta) \qquad b(\omega) = \sinh(\omega + \eta/2)$$

$$c(\omega) = \sinh(\omega - \eta/2) \qquad d(\omega) = \sqrt{\sinh\eta \sinh 2\eta}.$$

This matrix acts on the tensor product of the vector space $V_\sigma \otimes V_s$, that is a (2×3)-dimensional space. We first note that R^{σ_s} can be written as

$$R^{\sigma_s}(\omega) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \tag{31}$$

where

$$\begin{aligned} t_{11} &= f(\omega, \eta)s_3^2 + g(\omega, \eta)s_3 + h(\omega, \eta) \\ t_{22} &= f(\omega, \eta)s_3^2 - g(\omega, \eta)s_3 + h(\omega, \eta) \\ t_{12} &= ds_- \qquad t_{21} = ds_+ \end{aligned} \tag{32}$$

with s_3, s_+, s_- representing the spin-1 matrix operators of $SU(2)$:

$$s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad s = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{33}$$

where $s_\pm = s_1 \pm \mathrm{i}s_2$ and the functions f, g, h are given as

$$\begin{aligned} f(\omega, \eta) &= \tfrac{1}{2}(a - 2b + c) \\ g(\omega, \eta) &= \tfrac{1}{2}(a - c) \\ h(\omega, \eta) &= b. \end{aligned} \tag{34}$$

If we expand f, g, h around $\omega = \infty$ then we obtain

$$\begin{aligned} f &\xrightarrow[\omega\to\infty]{} \mathrm{e}^{\omega+\eta/2} \sinh^2 \eta/2 \\ g &\xrightarrow[\omega\to\infty]{} \tfrac{1}{2}\mathrm{e}^{\omega+\eta/2} \sinh \eta \\ h &\xrightarrow[\omega\to\infty]{} \tfrac{1}{2}\mathrm{e}^{\omega+\eta/2}. \end{aligned} \tag{35}$$

We will now utilise these to obtain the asymptotic form for the elements of T^{σ_s}. Since

$$T^{\sigma_s}_{ab}(\omega, \{\mu\}) = \sum_{a_1,\dots,a_{N-1}}^{2} t^{(1)}_{aa_1}(\omega + \mu_1)t^{(2)}_{a_1a_2}(\omega + \mu_2) \dots t^{N}_{a_{N-1}b}(\omega + \mu_N) \tag{36}$$

using (30) we at once obtain

$$\begin{aligned} A^{\sigma_s}(\omega, \{\mu\}) &\cong y^N(\omega)\left[\exp\left[\eta \sum_{i=1}^{N} s_3^i\right] + \frac{Q_+(\mu)}{y^2} + \mathrm{O}(y^{-4})\right] \\ D^{\sigma_s}(\omega, \{\mu\}) &\cong y^N(\omega)\left[\exp\left[-\eta \sum_{i=1}^{N} s_3^i\right] + \frac{Q_-(\mu)}{y^2} + \mathrm{O}(y^{-4})\right] \end{aligned} \tag{37}$$

$$\begin{aligned} B^{\sigma_s}(\omega, \{\mu\}) &\cong y^{N-1}(\omega)d\sum_{k=1}^{N} \mathrm{e}^{\bar{\mu}-\mu_k}\Sigma_-^k \\ C^{\sigma_s}(\omega, \{\mu\}) &\cong y^{N-1}(\omega)d\sum_{k=1}^{N} \mathrm{e}^{\bar{\mu}-\mu_k}\Sigma_+^k \end{aligned} \tag{38}$$

where

$$\Sigma_-^k = \exp\left[\eta\sum_{i=1}^{k-1} s_3^i\right] s_-^k \exp\left[-\eta\sum_{i=k+1}^{N} s_3^i\right]$$
$$\Sigma_+^k = \exp\left[-\eta\sum_{i=1}^{k-1} s_3^i\right] s_+^k \exp\left[\eta\sum_{i=k+1}^{N} s_3^i\right] \tag{39}$$

and

$$Q_\pm(\mu) = d^2 e^{2\bar\mu} \sum_{i\leqslant j<k\leqslant N} e^{-\mu_j-\mu_k} \exp\left[\pm\eta\sum_{i=1}^{j-1} s_3^i\right] s_\mp^j \exp\left[\mp\eta\sum_{m=j+1}^{k-1} s_3^m\right]$$
$$\times\, s_\pm^k \exp\left[\pm\eta\sum_{l=k+1}^{N} s_3^l\right]$$
$$d = \sqrt{\sinh\eta\sinh 2\eta} \tag{40}$$
$$y = \tfrac{1}{2} e^{\omega+\eta/2+\bar\mu}$$
$$\mu = \frac{1}{N}\sum_{k=1}^{N}\mu_k \qquad \Sigma_3 = \sum_i s_3^i.$$

Substituting these expressions in $t^s(\omega, \{\mu\})$ we obtain

$$t^s(\omega,\{\mu\}) = y^{2N}(\omega)\bigg[(e^{2\eta\Sigma_3}+e^{-2\eta\Sigma_3}+1)+\frac{1}{y^2}\{(e^{\eta}Q_+(\mu)e^{\eta\Sigma_3}+e^{-\eta}e^{\eta\Sigma_3}Q_+(\mu)+e^{\eta}Q_-e^{\eta\Sigma_3}$$
$$+\,e^{-\eta}e^{-\eta\Sigma_3}Q_-(\mu)+e^{\eta}Q_-(\mu)e^{-\eta\Sigma_3}+e^{-\eta}e^{\eta\Sigma_3}Q_-(\mu))\}$$
$$+\,d^2\sum_{k,l=1}^{N} e^{2\bar\mu-\mu_k-\mu_l}(\Sigma_-^k-\Sigma_-^l)\bigg]. \tag{41}$$

Extracting the coefficient of y^{2N-2} gives the Hamiltonian

$$H_{2N-2} = e^{\eta}\{Q_+(\mu)e^{\eta\Sigma_3}+Q_-(\mu)e^{-\eta\Sigma_3}+Q_+(\mu)e^{-\eta\Sigma_3}\}$$
$$+\,e^{-\eta}\{e^{\eta\Sigma_3}Q_+(\mu)+e^{\eta\Sigma_3}Q_-(\mu)+e^{-\eta\Sigma_3}Q_-(\mu)\}$$
$$+\,d^2\sum_{k,l=1}^{N} e^{2\bar\mu-\mu_k-\mu_l}(\Sigma_-^k\Sigma_+^l) \tag{42}$$

which is actually a long-range Hamiltonian coupling of the spin-1 operators s_3, s_+, s_- at different lattice sites. It was the simple observation of de Vega [4] that the expression of the transfer matrix in the parameter y leads to various long-range Hamiltonians. We have applied it in the case of the fused (doubly) t-matrix $t^s(\omega, \mu)$.

5. Eigenvalues

The eigenvalues pertaining to this Hamiltonian can be extracted from our general expression given in equation (29), by the corresponding expansion in the variable y.

If we set

$$x = \tfrac{1}{2}e^{u+\bar{\mu}+\eta/2}$$

then after a simple calculation we obtain

$$\begin{aligned} \delta(u) &\cong x^N e^{-N\eta/2}\left[1 - \frac{e^\eta}{4x^2}\sum_{k=1}^{N} e^{2(\bar{\mu}-\mu_k)} + \cdots\right] \\ \alpha(u) &\cong x^N e^{N\eta/2}\left[1 - \frac{1}{4x^2}\sum_{k=1}^{N} e^{2(\bar{\mu}-\mu_k)} + \cdots\right]. \end{aligned} \tag{43}$$

Substituting in equation (29) we obtain

$$\begin{aligned} \tilde{E} = &\left\{R_1 S_1 (e^{2\eta}+1) + \frac{R_1^2}{4}(e^\eta + e^{-3\eta})\sum_K e^{2(\bar{\mu}-\mu_k)}\right\} \\ &+ \left\{R_1 S_1 (1+e^{-2\eta}) + \frac{R_1^2}{4}(e^{-\eta} + e^{3\eta})\sum_k e^{2(\bar{\mu}-\mu_k)}\right\} \\ &- \{R_1^3 S_1(1+e^{-2\eta}) + R_1^3 S_1 (e^{2\eta}+1)\} \end{aligned} \tag{44}$$

where

$$\begin{aligned} R_1 &= e^{\eta/2(N-2m)} + e^{-\eta/2(N-2m)} \\ S_1 &= e^{\eta/2(N-2m)}\left\{\sum_{k=1}^{N} e^{2(\bar{\mu}-\mu_k)} - 2\sinh\eta \sum_{i=1}^{m} e^{2(\bar{\mu}+v_i)}\right\} \\ &\quad + e^{-\eta/2(N-2m)}\left\{e^\eta \sum_{k=1}^{N} e^{2(\bar{\mu}-\mu_k)} - 2\sinh\eta\sum_{i=1}^{m} e^{2(\mu+v_i)}\right\} \end{aligned} \tag{45}$$

which is the required eigenvalue corresponding to the long-range Hamiltonian (42).

6. Bethe ansatz equations

In the above analysis we still do not know how the quasi-momenta v_i are determined; the eigenvalues and many other quantities are determined by them. The equations determining v_i can be obtained by demanding that the residue at the poles of the exact expression for $E(\theta, \{\mu\})$ will vanish. We go back to equations (29), (15) and (13) and evaluate the residue at $v = v_k$. The result of this calculation can be presented in a compact form if we define

$$A(\theta) = \sinh(\theta - v_j - \eta)\prod_{\substack{i=1\\ i\neq j}}^{m}\frac{\sinh(\theta - v_i - \eta)}{\sinh(\theta - v_i)}\alpha(\theta) + \sinh(\theta - v_j + \eta)\prod_{\substack{i=1\\ i\neq j}}^{m}\frac{\sinh(\theta - v_i + \eta)}{\sinh(\theta - v_i)}\delta(\theta) \tag{46}$$

$$\begin{aligned} Q &= E^\sigma(\theta-\eta)A(\theta)d(\theta+\eta/2) \\ R &= E^\sigma(\theta+\eta)A(\theta)d(\theta-\eta/2) \\ P &= E^\sigma(\theta-\eta)A^2(\theta)E^\sigma(\theta+\eta). \end{aligned} \tag{47}$$

The Bethe ansatz equation can now be written as

$$\mathrm{Res}_{\theta=v_j}[E^s(\theta,\{\mu\})] = \mathrm{Res}_{\theta=v_j}\left[\frac{P}{\sinh^2(\theta-v_j)} - \frac{Q+R}{\sinh(\theta-v_j)}\right] \tag{48}$$

which, on simplification, leads to

$$\prod_{K=1}^{N}\frac{\sinh(v_j+\mu_K+\eta)}{\sinh(v_j+\mu_K)} = \prod_{\substack{i=1\\ i\neq j}}^{m}\frac{\sinh(v_i-v_j-\eta)}{\sinh(v_i-v_j+\eta)} \tag{49}$$

which is very similar to the case of the spin-1 open chain considered by Mezincescu and Nepomachia with nearest neighbour interaction [3].

7. Conclusions

In our analysis we have shown how, with the help of the fusion procedure, one can construct a long-range spin-1 chain and solve it completely by using the commutativity of the transfer matrix for the pure spin-1/2 case and that of the mixed $(1/2-1)$ system, and also with the transfer matrix of the spin-1 system. The long-range Hamiltonian has been written solely in terms of spin-1 operators.

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